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In memory of S.I. Pohozaev

On the Solvability of Some Nonlinear Elliptic Problems

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Abstract—The solvability of second-order nonlinear elliptic equations in weighted Sobolev spaces is analyzed. An additional condition ensuring the solvability of such equations is that the average of the desired solution over some circle of fixed radius is zero. Examples are equations containing a weighted p-Laplacian and the Euler equations.

Keywords: nonlinear weighted elliptic problems, solvability, average of the solution.

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1. INTRODUCTION

In the entire Euclidean space, we consider nonlinear weighted elliptic equations, in particular, their typical representatives

$$\mathfrak{A}_p u \equiv -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(w(|x|)\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) = h(x), \quad x \in \mathbb{R}^n$$

(where $\mathfrak{A}_p$ is a nonlinear weighted Laplacian) and

$$\Delta_p u \equiv \operatorname{div}(w(|x|)|\nabla u|^{p-2} \nabla u) = h(x), \quad x \in \mathbb{R}^n$$

(where $\Delta_p u$ is a weighted p-Laplacian).

Both operators are heavily used in the elliptic and parabolic theories of second-order nonlinear differential equations (see, e.g., [1, 2]).

In a bounded domain, the solvability of corresponding boundary value problems is a well-known issue that can generally be regarded as completely studied.

In the case of the complete Euclidean space $\mathbb{R}^n$, the situation is different, which is associated primarily with the lack of the Friedrichs and Poincaré inequalities; i.e., the norm of a function cannot be estimated in terms of its gradient in classical norms of Sobolev spaces.

Numerous well-known methods for studying quasilinear elliptic problems in $\mathbb{R}^n$ (specifically, classical and special variational methods and the method of monotone operators) can be found in [3], which deals with equations of the form

$$-\Delta u + f(x,u) = 0, \quad x \in \mathbb{R}^n,$$

where $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition with the additional condition

$$\lim_{|x|\to\infty} u(x) = 0.$$

A weak solution is sought in the closure of the set of infinitely differentiable compactly supported functions $D(\mathbb{R}^n)$ with respect to the norm

$$\|\nabla \cdot\|_2 + \|\cdot\|_p + \|\cdot\|_q, \quad 1 \le p \le q < \infty.$$

These methods are also applicable to the more general quasilinear equations

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + f(x,u), \quad x \in \mathbb{R}^n,$$

with p-Laplacian in the principal part.

Over the last decade, much attention has also been given to p-Laplacian equations defined in the entire space.

For example, [4] is concerned with the equation

$$\Delta_p u = g(h,u), \quad x \in \mathbb{R}^n,$$

where $1 < p < n$ and the function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. Specifically, existence and blow-up theorems are proved for nontrivial nonnegative weak solutions $u \in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ or $u \in D^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\lim_{|x|\to\infty} u(x) = 0.$$

The regularity and qualitative properties of solutions are established.

The existence and uniqueness of a positive radially symmetric solution of the problem

$$-\Delta_p u = \rho(x) f(u), \quad x \in \mathbb{R}^n,$$
$$\lim_{|x|\to\infty} u(x) = 0$$

in the class $C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n\backslash\{0\})$, where $1 < p < \infty$ and $n \geq 3$, are analyzed in [5].

In [6] the existence and regularity of nontrivial positive weak solutions are investigated for the class of elliptic problems

$$-\Delta_p u + a(x)|u|^{p-2}u = f(x)|u|^{\alpha-1}u, \quad x \in \mathbb{R}^n,$$
$$\lim_{|x|\to\infty} u(x) = 0,$$

where $1 < p < n$ and α is a real constant such that $0 < \alpha < p^* - 1$ with $p^* = \dfrac{np}{n-p}$. The weak solutions belong to the class

$$\left\{ u \in W^1_p(\mathbb{R}^n) \middle| \int_{\mathbb{R}^n} (|\nabla u|^p + a(x)|u|^p\, dx < \infty \right\}.$$

The equation

$$-\Delta_p u + V(x)|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^n,$$
$$u \in W^1_p(\mathbb{R}^n),$$

where $1 < p < n$, is studied in [7]. It is shown that the energy functional

$$\Phi(u) = \frac{1}{p}\int_{\mathbb{R}^n} (|\nabla u|^p + V(x)|u|^p)dx - \int_{\mathbb{R}^n} F(x,u)dx$$

has critical points, i.e., the equation has a weak solution.

Equations with p-Laplacian are also widely used in eigenvalue problems.

In this paper, the solvability of nonlinear weighted equations of the indicated form is analyzed on the entire plane $\mathbb{R}^2_{x_1,x_2}$ without making any a priori assumptions about the asymptotic behavior of the desired solution as $|x| \to \infty$. However, we impose the additional condition that the average integral value of the solution over some circle of radius $R > 0$ is equal to zero. In the presence of gradient estimates in the entire space, this condition makes it possible to estimate the function itself with a corresponding weight. It turns out that this is sufficient for the unique solvability of the problem.

First, however, we need weighted inequalities similar to the classical Hardy and Poincaré ones.

2. TWO-SIDED HARDY INEQUALITIES

Let $w(r)$ be a measurable almost everywhere positive function on an arbitrary interval (a, b), where $0 \le a < b \le +\infty$. In what follows, $p > 1$ is an arbitrary number and $p' = p/(p-1)$.

The task is, given an arbitrary number $R \in (a, b)$, to find a nonnegative function $v_R(r)$ such that

$$\int_a^b v_R(r)\left|\int_R^r f(t)dt\right|^p dr \le M\int_a^b |f(r)|^p w(r)dr, \tag{1}$$

where $f(r)$ is an arbitrary measurable function on (a, b) for which the integral on the right-hand side of (1) converges and the constant $M > 0$ is independent of R.

Remark 1. Since $f(r)$ is arbitrary, (1) is equivalent to two Hardy inequalities:

$$\int_a^R v_R(r)\left|\int_R^r f(t)dt\right|^p dr \le M\int_a^R |f(r)|^p w(r)dr$$

(reverse Hardy inequality) and

$$\int_R^b v_R(r)\left|\int_R^r f(t)dt\right|^p dr \le M\int_R^b |f(r)|^p w(r)dr$$

(direct Hardy inequality). We will use both of them.

Theorem 1. *Suppose that $s = 1/(p-1)$ and w^{-s} belongs to $L_1^{\text{loc}}(a, b)$, i.e., w^{-s} is Lebesgue integrable in a neighborhood of any point of (a, b). Then the weight function $v_R(r)$ can be defined as*

$$v_R(r) = \mu_r(r) + pC_a\delta(r-a) + pC_b\delta(b-r),$$

where

$$\mu_R(r) = \left|\int_R^r w^{-s}(t)dt\right|^{-p} w^{-s}(r), \quad r \ne R;$$

the constants C_a and C_b are given by

$$C_a = \begin{cases} 0, & \displaystyle\int_a^R w^{-s}(t)dt = +\infty, \\ s\left(\displaystyle\int_a^R w^{-s}(t)dt\right)^{-1/s}, & \displaystyle\int_a^R w^{-s}(t)dt < +\infty, \end{cases}$$

$$C_b = \begin{cases} 0, & \displaystyle\int_R^b w^{-s}(t)dt = +\infty, \\ s\left(\displaystyle\int_R^b w^{-s}(t)dt\right)^{-1/s}, & \displaystyle\int_R^b w^{-s}(t)dt < +\infty; \end{cases}$$

$\delta(r)$ is the Dirac delta function; and $M = [p/(p-1)]^p$. In other words,

$$\int_a^b \mu_R(r)\left|\int_R^r f(t)dt\right|^p dr + pC_a\left|\int_a^R f(t)dt\right|^p + pC_b\left|\int_R^b f(t)dt\right|^p \le \left(\frac{p}{p-1}\right)^p\int_a^b |f(r)|^p w(r)dr. \tag{2}$$

Proof. We need the following result.

Lemma 1. *It holds that*

$$\mu_R(r) = \frac{1}{p-1}\nu'_R(r), \quad \mu_R(r) = |\nu_R(r)|^{p'} w^{-s}(r)$$

on (a, R) *and* (R, b), *where*

$$\nu_R(r) = \left|\int_R^r w^{-s}(t)dt\right|^{-p+1} \operatorname{sgn}(R-r)$$

and

$$\lim_{\varepsilon\to 0} \nu_R(R\pm\varepsilon)\left|\int_R^{R\pm\varepsilon} f(t)dt\right|^p = 0.$$

Proof. Obviously, only the last equality has to be checked. Indeed, using the Hölder inequality and recalling the definition of $\nu_R(r)$, we obtain

$$|\nu_R(R\pm\varepsilon)|\left|\int_R^{R\pm\varepsilon} f(t)dt\right|^p \le |\nu_R(R\pm\varepsilon)|\left|\int_R^{R\pm\varepsilon} |f(t)|^p w(t)dt\right|\left|\int_R^{R\pm\varepsilon} w^{-s}(t)dt\right|^{p-1} = \int_R^{R\pm\varepsilon} |f(t)|^p w(t)dt \to 0$$

as $\varepsilon \to 0$.

Now, we will prove inequality (2). For example, consider the case of (a, R).

By Lemma 1,

$$J \equiv \int_a^R \mu_R(r)\left|\int_r^R f(t)dt\right|^p dr = \frac{1}{p-1}\int_a^R \nu'_R(r)\left|\int_r^R f(t)dt\right|^p dr$$

$$= \frac{1}{p-1}\left[\nu_R(r)\left|\int_r^R f(t)dt\right|^p\right]\Bigg|_{r=a}^{r=R} + \frac{p}{p-1}\int_a^R \nu_R(r)\left|\int_r^R f(t)dt\right|^{p-1} \operatorname{sgn}\left(\int_r^R f(t)dt\right) f(r)dr$$

$$\le -\frac{1}{p-1}\nu_R(a)\left|\int_a^R f(t)dt\right|^p + \frac{p}{p-1}\int_a^R \nu_R(r) w^{-1/p}(r)\left|\int_r^R f(t)dt\right|^{p-1} |f(r)| w^{1/p}(r)dr.$$

Combining this result with the Hölder and Young inequalities yields

$$J + C_a\left|\int_a^R f(t)dt\right|^p \le \frac{p}{p-1}J^{1/p'}\left(\int_a^R |f(t)|^p w(r)dr\right)^{1/p} \le \frac{1}{p'}J + \frac{p^{p-1}}{(p-1)^p}\int_a^R |f(r)|^p w(r)dr,$$

i.e., we obtain the desired inequality

$$\int_a^R \mu_R(r)\left|\int_r^R f(t)dt\right|^p dr + pC_a\left|\int_a^R f(t)dt\right|^p \le \left(\frac{1}{p-1}\right)^p \int_a^R |f(r)|^p w(r)dr.$$

The case of the interval (R, b) is treated in a completely similar manner. The theorem is proved.

Example 1. Let $a = 0$, $b = \infty$, and $w(r) = r^{p-1}$. Then $\mu_R(r) = 1/\left(r\left|\ln\frac{r}{R}\right|^p\right)$ and $C_a = C_b = 0$; hence, inequality (2) becomes (see [8])

$$\int_0^\infty \frac{1}{r\left|\ln\frac{r}{R}\right|^p}\left|\int_R^r f(t)dt\right|^p dr \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(r)|^p r^{p-1}dr.$$

Example 2 (cf. [9]). Let $q \neq p$ be an arbitrary number, $a = 0$, $b = \infty$, and $w(r) = r^{q-1}$. Then $\mu_R(r) = \left|R^{-\alpha+1} - r^{-\alpha+1}\right|^{-p} r^{-\alpha}$, where $r \neq R$, $\alpha = (q-1)/(p-1)$, and

$$C_a = \begin{cases} 0, & q > p, \\ \dfrac{(p-q)^{p-1}}{(p-1)^p} R^{q-p}, & q < p, \end{cases} \qquad C_b = \begin{cases} 0, & q < p, \\ \dfrac{(q-p)^{p-1}}{(p-1)^p} R^{q-p}, & q > p. \end{cases}$$

Thus, we have

$$\int_0^R \mu_R(r)\left|\int_r^R f(t)dt\right|^p dr + p\frac{(p-q)^{p-1}}{(p-1)^p}R^{q-p}\left|\int_0^R f(t)dt\right|^p \le \left(\frac{1}{p-1}\right)^p \int_0^R |f(r)|^p r^{q-1} dr$$

if $q < p$ and

$$\int_R^\infty \mu_R(r)\left|\int_R^r f(t)dt\right|^p dr + p\frac{(q-p)^{p-1}}{(p-1)^p}R^{q-p}\left|\int_R^\infty f(t)dt\right|^p \le \left(\frac{p}{p-1}\right)^p \int_R^\infty |f(r)|^p r^{q-1} dr$$

if $q > p$.

Remark 2. Note that the regular part of the weight function $v_R(r)$ (i.e., $\mu_R(r)$) is integrable at both $r = a$ and $r = b$. Thus, the measure $\mu_R(r)dr$ compactifies the real line. This is important in deriving Poincaré-type inequalities for functions defined in the entire plane.

3. POINCARÉ INEQUALITIES IN THE PLANE

Let $x = (x_1, x_2)$ and $y \in L_1^{\text{loc}}(\mathbb{R}^2)$. The gradient $\nabla u(x)$ of $u(x)$ is understood in the sense of generalized functions (distributions). Assume that $\nabla u(x)$ is pth-power integrable in $\mathbb{R}^2$ ($p > 1$) with weight $w(|x|)$, i.e.,

$$\int_{\mathbb{R}^2} |\nabla u(x)|^p w(|x|)dx < \infty.$$

Introducing the weight function $w_2(r) = w(r)r$, where $r = |x|$, we assume that $w_2^{-s} \in L_1^{\text{loc}}(\mathbb{R}_+)$. Furthermore, let

$$\mu_{R,2}(r) = \left|\int_R^r w_2^{-s}(t)dt\right|^{-p} w_2^{-s}(r), \quad r \neq R,$$

be the "canonical" weight defined in Theorem 1.

Theorem 2. *Let*

$$\int_{S_{R,2}} u(x)ds = 0, \tag{3}$$

i.e., the average value of $u(x)$ *over the circle* $S_{R,2} \equiv \{x \in \mathbb{R}^2 : |x| = R\}$ *is zero. Then*

$$\int_{\mathbb{R}^2} \nu_{R,2}(|x|)|u(x)|^p dx \le M \int_{\mathbb{R}^2} |\nabla u(x)|^p w(|x|)dx, \tag{4}$$

where $M > 0$ *is a constant and*

$$\nu_{R,2}(r) \equiv r^{-1}\min\{w_2(r)r^{-p}, \mu_{R,2}(r)\} = \min\{w(r)r^{-p}, \mu_{R,2}(r)r^{-1}\}.$$

Proof. Consider the integral

$$\int_{\mathbb{R}^2} \nu_{R,2}(|x|)|u(x)|^p dx = \int_0^\infty \nu_{R,2}(r)r\int_0^{2\pi} |u(r,\phi)|^p d\phi dr.$$

Using the Poincaré inequality for periodic functions yields

$$\int_{\mathbb{R}^2} \nu_{R,2}(|x|)|u(x)|^p\,dx \le M\int_0^\infty \nu_{R,2}(r)r\left|\int_0^{2\pi} u(r,\phi)d\phi\right|^p dr + M\int_0^\infty \nu_{R,2}(r)r\int_0^{2\pi}\left|\frac{\partial}{\partial\phi}u(r,\phi)\right|^p d\phi dr \equiv M(I_1+I_2).$$

First, we estimate the integral

$$I_1 = \int_0^\infty \nu_{R,2}(r)r\,|f(r)|^p dr,$$

where $f(r) = \int_0^{2\pi} u(r,\phi)d\phi$. Note that $\nu_{R,2}(r)r \le \mu_{R,2}(r)$. Moreover, by assumption, $f(R) = 0$. Therefore, according to (2), we have

$$\begin{aligned} I_1 &\le \int_0^\infty \mu_{R,2}(r)|f(r)|^p\,dr \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |f'(r)|^p\,w(r)rdr = \left(\frac{p}{p-1}\right)^p \int_0^\infty w(r)r\left|\int_0^{2\pi}\frac{\partial u}{\partial r}(r,\phi)d\phi\right|^p dr \\ &\le M\int_0^\infty w(r)r\int_0^{2\pi}|\nabla u(r,\phi)|^p\,d\phi dr = M\int_{\mathbb{R}^2}|\nabla u(x)|^p\,w(|x|)dx. \end{aligned} \tag{5}$$

Consider the second integral

$$I_2 = \int_0^\infty \nu_{R,2}(r)r\int_0^{2\pi}\left|\frac{\partial}{\partial\phi}u(r,\phi)\right|^p d\phi dr.$$

Since the Cartesian coordinates are linear in r, it holds that $\left|\frac{\partial}{\partial\phi}u\right|^p \le Mr^p|\nabla u|^p$. Therefore, in view of the inequality $\nu_{R,2}(r) \le w(r)r^{-p}$, we obtain

$$I_2 \le M\int_0^\infty \nu_{R,2}(r)r^{p+1}\int_0^{2\pi}|\nabla u|^p d\phi dr \le M\int_0^\infty w(r)r\int_0^{2\pi}|\nabla u|^p\,d\phi dr = M\int_{\mathbb{R}^2}|\nabla u|^p\,w(|x|)dx.$$

Combining this relation with (5) yields the desired inequality (4).

Remark 3. The constant $M > 0$ in (4) depends only on $p > 1$.

Example 3. Let $w(x) = |x|^{p-2}$ and condition (3) be satisfied. Then

$$\int_{\mathbb{R}^2} \nu_{R,2}(|x|)|u(x)|^p\,dx \le M\int_{\mathbb{R}^2}|\nabla u(x)|^p\,|x|^{p-2}\,dx,$$

where

$$\nu_{R,2}(r) = \min\left\{r^{-2}\left|\ln\frac{r}{R}\right|^{-p}, r^{-2}\right\}, \tag{6}$$

i.e.,

$$\nu_{R,2}(r) = \begin{cases} r^{-2}\left|\ln\frac{r}{R}\right|^{-p}, & r \in \left(0,\frac{R}{e}\right)\cup(\mathrm{Re},+\infty), \\ r^{-2}, & r \in \left[\frac{R}{e},\mathrm{Re}\right]. \end{cases}$$

Example 4. Let $u \in L_1^{\mathrm{loc}}(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2}|\nabla u(x)|^p\,|x|^{p-2}\,dx < \infty, \quad q \in \mathbb{R}.$$

Additionally, let condition (3) be satisfied. Then

$$\int_{\mathbb{R}^2} \nu_{R,2}(|x|)|u(x)|^p\,dx \le M\int_{\mathbb{R}^2} |\nabla u(x)|^p |x|^{q-2}\,dx,$$

where $\nu_{R,2}(r) = r^{-1}\min\{\mu_R(r), r^{q-2-p}\}$ and $\mu_R(r)$ is given by the formula

$$\mu_{R,2} = |\alpha - 1|^p \left|R^{-\alpha+1} - r^{-\alpha+1}\right|^{-p} r^{-\alpha}, \quad r \ne R, \quad \alpha = (q-1)/(p-1)$$

if $q \ne p$ and by formula (6) if $q = p$.

Consequences of (4) are the Poincaré-type inequalities

$$\int_{\mathbb{R}^2} \nu_{R,2}(|x|)|u(x) - C|^p\,dx \le M\int_{\mathbb{R}^2} |\nabla u(x)|^p w(|x|)dx,$$

where

$$C = \frac{1}{\operatorname{mes} S_{R,2}} \int_{S_{R,2}} u(x)ds$$

and

$$\int_{\mathbb{R}^2} \nu_{R,2}(|x|)|u(x)|^p\,dx \le M\left(\int_{\mathbb{R}^2} |\nabla u(x)|^p w(|x|)dx + \left|\int_{S_{R,2}} u(x)ds\right|^p\right).$$

4. FORMULATION OF THE PROBLEM, THE BASIC SPACES, AND SOME INEQUALITIES

The problem is to find a solution of the equation

$$\mathfrak{A}_p u \equiv -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(w(|x|)\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) = h, \quad x \in \mathbb{R}^n, \tag{7}$$

satisfying the condition

$$\int_{S_R} u(x)ds = 0, \tag{8}$$

where $p > 1$ and $n = 2$.

Condition (8) means that the function u has a zero average value over some circle S_R of radius R. The problem for the p-Laplacian is set up in a similar manner.

We seek a weak solution of problem (7), (8), specifically, a function $u(x)$ with a zero average value over S_R that satisfies the integral identity

$$\int_{\mathbb{R}^2} \sum_{i=1}^{2} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} w(|x|)dx = \int_{\mathbb{R}^2} h(x)v(x)dx,$$

where $v(x)$ runs over a suitable Sobolev-type space.

Let $w(r)$ be a measurable almost everywhere positive function on $(0, +\infty)$. Suppose that $u \in L_1^{\mathrm{loc}}(\mathbb{R}^2)$ satisfies condition (8) and its gradient ∇u belongs to the weighted Lebesgue space $L_{p,\,w}(\mathbb{R}^2)$, i.e.,

$$\int_{\mathbb{R}^2} |\nabla u(x)|^p w(|x|)dx < \infty.$$

Then, by Theorem 2,

$$\int_{\mathbb{R}^2} |u(x)|^p \nu_{R,2}(|x|)dx \le M\int_{\mathbb{R}^2} |\nabla u(x)|^p w(|x|)dx, \tag{$*$}$$

where

$$\nu_{R,2}(r) \equiv r^{-1}\min\{w_2(r)r^{-p}, \mu_{R,2}(r)\} = \min\{w(r)r^{-p}, \mu_{R,2}(r)r^{-1}\}$$

and

$$\mu_{R,2}(r) = \left|\int_R^r w_2^{-s}(t)dt\right|^{-p} w_2^{-s}(r), \quad r \neq R,$$

is the canonical weight defined in Theorem 1; here, $w_2(r) = w(r)r$, $w_2^{-s} \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$, and $s = 1/(p-1)$. By using the norms in the weighted spaces, inequality (*) can be rewritten the following form (the notation is clear):

$$\|u\|^p_{L_{p,\nu}(\mathbb{R}^2)} \le M\|\nabla u\|^p_{L_{p,w}(\mathbb{R}^2)}. \tag{**}$$

Let $V^1_{p,\nu,w}(\mathbb{R}^2)$ denote the space of functions

$$\left\{u : \mathbb{R}^2 \to \mathbb{R}^1 | u \in L_{p,\nu}(\mathbb{R}^2), \nabla u \in L_{p,w}(\mathbb{R}^2), \int_{S_R} u(x)ds = 0\right\},$$

which is hereafter designated for brevity as X, and let its norm be given by

$$\|\cdot\|_X \equiv \|\nabla\cdot\|_{L_{p,w}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |\nabla\cdot|^p w(|x|)dx\right)^{\frac{1}{p}}$$

or, equivalently,

$$\|\cdot\|'_X \equiv \|\nabla\cdot\|_{L_{p,w}(\mathbb{R}^2)} + \|\cdot\|_{L_{p,\nu}(\mathbb{R}^2)}.$$

It easy to show that X is a separable reflexive Banach space. Another norm $\|\cdot\|$ in X can be defined by the formula

$$\|u\| = \left(\int_{\mathbb{R}^2}\sum_{i=1}^2 \left|\frac{\partial u}{\partial x_i}\right|^p w(|x|)dx\right)^{\frac{1}{p}}.$$

Since $w(|x|) > 0$ for $x \in \mathbb{R}^2$, we have

$$2^{-1}\|u\|^p \le \|u\|^p_X \le 2^{\frac{p-1}{2}}\|u\|^p,$$

so this norm is equivalent to the basic norm $\|\cdot\|_X$.

Let an operator $A : X \to X^*$ be defined by the formula

$$\langle Au, v\rangle = \int_{\mathbb{R}^2}\sum_{i=1}^2 \left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}w(|x|)dx, \tag{9}$$

where $u \in X$, $v \in X$, and X^* is the dual of X.

Clearly,

$$\|Au\|_{X^*} = \sup_{v\in X, v\neq 0}\frac{|\langle Au, v\rangle|}{\|v\|_X} \le 2\|u\|^{p-1}_X. \tag{10}$$

Lemma 2. *For any* u, $v \in X$,

$$\langle Au - Av, u - v\rangle \ge 2^{-\frac{p-1}{2}}c\|u - v\|^p_X, \tag{11}$$

where $c > 0$ *is a constant.*

Proof. By definition (9) of the operator $A : X \to X^*$,

$$\langle Au - Av, u - v\rangle = \int_{\mathbb{R}^2} \sum_{i=1}^{2} \left[\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v}{\partial x_i} \right] \frac{\partial}{\partial x_i}(u - v) w(|x|) dx.$$

Let $\xi_i = (\partial u)/(\partial x_i)$, $\eta_i = (\partial v)/(\partial x_i)$, $F(\xi) = |\xi|^{p-2}\xi$, and $\zeta(\tau) = \eta + \tau(\xi - \eta)$, where $0 \le \tau \le 1$. We have

$$F'(\zeta) = (p-2)|\zeta|^{p-3} \operatorname{sgn} \zeta\zeta + |\zeta|^{p-2} = (p-1)|\zeta|^{p-2}$$

and, for $\zeta \neq 0$,

$$F(\xi) - F(\eta) = F(\eta + \tau(\xi - \eta))\big|_{r=1} - F(\eta + \tau(\xi - \eta))\big|_{r=0} = \int_0^1 \frac{dF}{d\tau}(\zeta(\tau)) d\tau$$

$$= \int_0^1 F'(\zeta)\zeta'(\tau) d\tau = \int_0^1 (p-1)|\eta + \tau(\xi - \eta)|^{p-2} (\xi - \eta) d\tau.$$

Then

$$\langle Au - Av, u - v\rangle = \int_{\mathbb{R}^2} \sum_{i=1}^{2} (F'(\xi_i) - F(\eta_i))(\xi_i - \eta_i) w(|x|) dx$$

$$= (p-1) \int_{\mathbb{R}^2} \sum_{i=1}^{2} \int_0^1 |\eta_i + \tau(\xi_i - \eta_i)|^{p-2} d\tau (\xi_i - \eta_i)^2 w(|x|) dx.$$

For arbitrary numbers a and b, it is true that

$$\int_0^1 |a + \tau b|^{p-2} d\tau \ge c|b|^{p-2},$$

where $c > 0$ is a constant. Using this inequality, we obtain

$$\langle Au - Av, u - v\rangle \ge (p-1)c \int_{\mathbb{R}^2} \sum_{i=1}^{2} |\xi_i - \eta_i|^{p-2} (\xi_i - \eta_i)^2 w(|x|) dx$$

$$= c \int_{\mathbb{R}^2} \sum_{i=1}^{2} \left|\frac{\partial}{\partial x_i}(u - v)\right|^p w(|x|) dx \ge 2^{-\frac{p-1}{2}} c \int_{\mathbb{R}^2} \left| \sum_{i=1}^{2} \left(\frac{\partial}{\partial x_i}(u - v)\right)^2 \right|^{\frac{p}{2}} w(|x|) dx = 2^{-\frac{p-1}{2}} c \|u - v\|_X^p.$$

Now consider the case $\zeta = 0$, i.e., $\xi = \alpha\eta$, where $\alpha \neq 1$ is a number (for $\alpha = 1$, inequality (11) is obviously valid). We have

$$\langle Au - Av, u - v\rangle = \int_{\mathbb{R}^2} \sum_{i=1}^{2} (|\alpha|^{p-2}\alpha - 1)(\alpha - 1)|\eta_i|^2 w(|x|) dx$$

$$\ge (|\alpha|^{p-2}\alpha - 1)(\alpha - 1) 2^{-\frac{p-1}{2}} \|v\|_X^p = \frac{|\alpha|^{p-2}\alpha - 1}{(\alpha - 1)^{p-1}} 2^{-\frac{p-1}{2}} \|u - v\|_X^p.$$

Moreover, since the norms $\|\cdot\|_X$ and $\|\cdot\|$ are equivalent for any $u \in X$, it holds that

$$\langle Au, u\rangle \ge 2^{-\frac{p-1}{2}} \|u\|_X^p. \tag{12}$$

5. MAIN RESULTS

Definition. A function $u \in X$ is called a *weak solution of problem* (7), (8) with $h \in X^*$ if, for any function $v \in X$,

$$\langle Au, v \rangle = \langle h, v \rangle, \tag{13}$$

where A is an operator from X to X^* defined by formula (9). The symbol $\langle h, v \rangle$ denotes the result of applying the functional $h \in X^*$ to the function $v \in X$.

Theorem 3. *For any functional $h \in X^*$, problem* (7), (8) *has a unique weak solution in the sense of above definition.*

Proof. Let the system $\{v_j\}_{j=1}^{\infty} \subset X$ be complete in X. Let $X_N = \operatorname{span}\{v_1, \ldots, v_N\}$, $X_N \subset X$, and $\bigcup_{N=1}^{\infty} X_N$ be dense everywhere in X.

An approximate solution $u_N \in X_N$ has the form $u_N = \sum_{j=1}^{N} c_j v_j$ and satisfies the relation

$$\langle Au_N, v \rangle = \langle h, v \rangle \quad \forall v \in X_N. \tag{14}$$

The solvability of Eq. (14) is equivalent to the existence of a solution to the system

$$P_j(\mathbf{c}) \equiv \langle Au_N, v_j \rangle - \langle h, v_j \rangle = 0, \quad j = 1, \ldots, N, \quad \mathbf{c} = (c_1, \ldots, c_n).$$

We have

$$(P(\mathbf{c}), \mathbf{c}) = \sum_{j=1}^{N} P_j(\mathbf{c}) c_j = \langle Au_N, u_N \rangle - \langle h, u_N \rangle.$$

By the definition of the norm $\|\cdot\|_{X^*}$,

$$|\langle h, u_N \rangle| \le \|h\|_{X^*} \|u_N\|_X.$$

Combining this inequality with (12) yields

$$(P(\mathbf{c}), \mathbf{c}) \ge \left(2^{-\frac{p-1}{2}} \|u_N\|_X^{p-1} - \|h\|_{X^*} \right) \|u_N\|_X.$$

It follows that $(P(\mathbf{c}), \mathbf{c}) > 0$ if $\|u_N\|_X$ is sufficiently large. Since all norms in a finite-dimensional space are equivalent, the inequality also holds for $|\mathbf{c}| = R$, where $R = R(h)$ is sufficiently large. Therefore, by the acute angle lemma, there exists at least one point $\mathbf{c}$ such that $P(\mathbf{c}) = 0$.

Setting $v = u_N \in X_N$ in (14), we have

$$2^{-\frac{p-1}{2}} \|u_N\|_X^p \overset{(12)}{\le} \langle Au_N, u_N \rangle \overset{(14)}{=} \langle h, u_N \rangle \le \|h\|_{X^*} \|u_N\|_X,$$

whence

$$\|u_N\|_X^{p-1} \le 2^{\frac{p-1}{2}} \|h\|_{X^*}. \tag{15}$$

Moreover, by virtue of the basic inequality (**), we have

$$\|u_N\|_{L_{p,v}(\mathbb{R}^2)} + \|u_N\|_X \le c_0 \|h\|_{X^*}^{\frac{1}{p-1}}, \tag{15'}$$

where $c_0 = \left(M^{\frac{1}{p}} + 1 \right) 2^{\frac{1}{2}}$.

Thus, the sequence $\{u_N\}_{N=1}^{\infty}$ is bounded in the reflexive space X and, hence, has a weakly limit point $u(x)$ in X. Without loss of generality, we can assume that

$$u_N \rightharpoonup u \quad \text{weakly in} \quad X \quad \text{as} \quad N \to \infty. \tag{16}$$

It follows from (10) and (15) that

$$\|Au_N\|_{X^*} \overset{(10)}{\leq} 2\|u_N\|_X^{p-1} \overset{(15)}{\leq} 2^{\frac{p+1}{2}} \|h\|_{X^*},$$

i.e., the sequence $\{Au_N\}_{N=1}^{\infty}$ is bounded in X^*.

The space X^* is reflexive as the dual of the reflexive space X. Therefore, we can assume (after choosing a subsequence) that $Au_N \rightharpoonup a \in X^*$ weakly in X^*.

Passing to the limit as $N \to \infty$ in (14) and recalling that the system $\{v_j\}_{j=1}^{\infty} \subset X$ is complete in X, we obtain

$$\langle a, v\rangle = \langle h, v\rangle \quad \forall v \in X,$$

i.e., $a = h$ in the sense of X^*.

Thus,

$$Au_N \rightharpoonup h \quad \text{weakly in} \quad X^* \quad \text{as} \quad N \to \infty. \tag{17}$$

We have $\forall v \in X$

$$\langle h, u_N\rangle - \langle Au_N v\rangle - \langle Av, u_N - v\rangle \overset{(14)}{=} \langle Au_N, u_N\rangle - \langle Au_N, v\rangle - \langle Av, u_N - v\rangle = \langle Au_N - Av, u_N - v\rangle \overset{(11)}{\geq} 0.$$

Combining this result with (16) and (17) and setting $N \to \infty$ yields

$$\langle h, u\rangle - \langle h, v\rangle - \langle Av, u - v\rangle = \langle h - Av, u - v\rangle \geq 0.$$

Let $v = u - \xi w$, where $w \in X$ is arbitrary and $\xi \to +0$. Then the last inequality implies that

$$\langle h - Av, u - v\rangle = \langle h - A(u - \xi w), \xi w\rangle = \xi\langle h - A(u - \xi w), w\rangle \xrightarrow{\xi \to +0} (+0)\langle h - Au, w\rangle \geq 0.$$

Since $w \in X$ is arbitrary, the last inequality holds only if $h = Au$ as elements of X^*. This means that (13) holds for any $v \in X$; thus, the found element $u \in X$ is a weak solution.

The uniqueness of u can be proved using inequality (11). Indeed, assume that there are two functions u_1, $u_2 \in X$ such that $Au_1 = h$ and $Au_2 = h$ in the sense of X^*. Then

$$\langle Au_1 - Au_2, u_1 - u_2\rangle \geq 2^{-\frac{p-1}{2}} c\|u_1 - u_2\|_X^p = 2^{-\frac{p-1}{2}} c\|\nabla(u_1 - u_2)\|_{L_{p,w}(\mathbb{R}^2)},$$

where $c > 0$ is a constant. Combining this with the condition $w(|x|) > 0$, $x \in \mathbb{R}^2$, we see that $\nabla(u_1 - u_2) = 0$, i.e., $u_1 - u_2 = \text{const}$ almost everywhere in $\mathbb{R}^2$. Since u_1 and u_2 satisfy condition (8), we obtain $u_1 = u_2$.

The theorem is proved.

A posteriori remark. Let us show that the sequence $\{u_N\}_{N=1}^{\infty} \subset X$ of approximate solutions converges to the weak solution $u \in X$ not only weakly, but also strongly.

Let $u \in X$ be the weak limit in (16). According to (11),

$$\langle Au_N - au, u_N - u\rangle \geq 2^{-\frac{p-1}{2}} c\|u_N - u\|_X^p.$$

On the left-hand side, by virtue of (14) with $v = u_N$, the equality $Au = h$ as elements of X^*, and relation (17), we have

$$\langle h, u_N\rangle - \langle Au, u_N\rangle - \langle Au_N, u\rangle + \langle h, u\rangle = -\langle Au_N, u\rangle + \langle h, u\rangle \xrightarrow{N \to \infty} -\langle h, u\rangle + \langle h, u\rangle = 0.$$

This means that

$$u_N \to u \quad \text{in} \quad X \quad \text{as} \quad N \to \infty. \tag{18}$$

6. EXAMPLE

Consider a right-hand side $h \in L_{p',\nu^{-1}}(\mathbb{R}^2)$, i.e.,

$$\|h\|_{p',\nu^{-1}(\mathbb{R}^2)}^{p'} = \int_{\mathbb{R}^2} |h(x)|^{p'} \nu^{-1}(|x|)dx < +\infty.$$

This function defines a linear bounded functional $h \in X^*$ given by the formula

$$\langle h, v\rangle = \int_{\mathbb{R}^2} h(x)\dot{v}(x)\nu^{\frac{2-p}{p}}(|x|)dx \quad \forall v \in X.$$

Indeed,

$$|\langle h, v\rangle| = \left|\int_{\mathbb{R}^2} h(x)\nu^{-\frac{1}{p'}}(|x|)v(x)\nu^{\frac{1}{p'}+\frac{2-p}{p}}(|x|)dx\right| \le \left(\int_{\mathbb{R}^2} |h(x)|^{p'}\nu^{-1}(|x|)dx\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^2}|v(x)|^p\nu(|x|)dx\right)^{\frac{1}{p}} = \|h\|_{p',\nu^{-1}(\mathbb{R}^2)}\|v\|_{L_{p,\nu}(\mathbb{R}^2)} \tag{19}$$

and (**) implies that

$$|\langle h, v\rangle| \le M^{\frac{1}{p}}\|h\|_{p',\nu^{-1}(\mathbb{R}^2)}\|v\|_X.$$

Then

$$\|h\|_{X^*} = \sup_{v\in X, v\ne 0}\frac{|\langle h, v\rangle|}{\|v\|_X} \le M^{\frac{1}{p}}\|h\|_{p',\nu^{-1}(\mathbb{R}^2)},$$

and it follows from (15') with (18) that

$$\|u\|_{L_{p,\nu}(\mathbb{R}^2)} + \|u\|_X \le c\|h\|_{p',\nu^{-1}(\mathbb{R}^2)}^{\frac{1}{p-1}}, \tag{20}$$

where $c = c_0 M^{\frac{1}{p(p-1)}}$.

Thus, for $h \in L_{p',\nu^{-1}}(\mathbb{R}^2)$, problem (7), (8) is well-posed; i.e., it has a unique weak solution $u_h \in X$ satisfying identity (13):

$$\langle Au_h, v\rangle = \langle h, v\rangle \quad \forall v \in X;$$

moreover, the a priori estimate (20) holds.

Let us show the necessity of the basic inequality (**) for the problem to be well-posed. We have

$$|\langle h, v\rangle| \overset{(13)}{\le} \|Au_h\|_{X^*}\|v\|_X \overset{(10)}{\le} 2\|u_h\|_X^{p-1}\|v\|_X \overset{(20)}{\le} 2c^{p-1}\|h\|_{p',\nu^{-1}(\mathbb{R}^2)}\|v\|_X,$$

whence

$$|\langle h, v\rangle| \le M_h\|v\|_X,$$

where $M_h > 0$ is a constant.

This inequality means that the set of linear functionals $\{l_v\}_{v\in X}$ acting on elements $h \in L_{p',\nu^{-1}}(\mathbb{R}^2)$ and given by the formula

$$l_v(h) \equiv \left\langle h, \frac{v}{\|v\|_X}\right\rangle,$$

where $v \in X$ is a parameter, is bounded for every $h \in L_{p',\nu^{-1}}(\mathbb{R}^2)$. Therefore, by the Banach–Steinhaus theorem, this set of functionals is norm bounded, i.e., there exists a constant $M > 0$ such that

$$\sup_{v\in X}\|l_v\| \le M.$$

Therefore, $\forall v \in X$

$$\|l_v\| \equiv \sup_{h\ne 0}\frac{\|l_v(h)\|}{\|h\|_{p',\nu^{-1}(\mathbb{R}^2)}} = \sup_{h\ne 0}\frac{|\langle h, v\rangle|}{\|h\|_{p',\nu^{-1}(\mathbb{R}^2)}\|v\|_X} \overset{(19)}{\le} \frac{\|v\|_{L_{p,\nu}(\mathbb{R}^2)}}{\|v\|_X} \le M,$$

whence

$$\|v\|_{L_{p,v}(\mathbb{R}^2)} \leq M\|v\|_X .$$

Thus, the following result holds in the case $h \in L_{p',v^{-1}}(\mathbb{R}^2)$.

Theorem 4. *The well-posedness of problem* (7), (8) *is equivalent to the validity of inequality* (**).

The general second-order nonlinear weighted equation

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\{w(|x|)A_i(x,u,\nabla u)\} = h(x)$$

is treated in an entirely similar fashion under the natural growth and coercivity conditions imposed on the functions A_i, $i = 1, 2$.

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